

Generalized Mountain Pass Lemma Related with a Closed Subset and Locally Lipschitz Functionals

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Abstract

The classical Mountain Pass Lemma of Ambrosetti-Rabinowitz has been studied, extended and modified in several directions; notable examples would certainly include the generalization to locally Lipschitz functionals in K.C. Chang, analysis of the structure of the critical set in the mountain pass theorem by Hofer and Pucci-Serrin and Tian, the extension by Ghoussoub-Preiss to closed subsets in a Banach space, and variations found in the recent Peral . In this paper, we utilize the generalized gradient of Clarke and Ekeland's variational principle to generalize the Ghoussoub-Preiss's Theorem in the setting of locally Lipschitz functionals.

Keywords: Mountain Pass Lemma of Ambrosetti-Rabinowitz, Ekeland's variational principle, locally Lipschitz functionals, Clarke's generalized gradient, generalized Mountain Pass Lemma.

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1 Introduction and Main Results

In 1973, Ambrosetti and Rabinowitz [1] published the famous Mountain-Pass Lemma:

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Theorem 1.1. ([1]) Let f be a C^1 -real functional defined on a Banach space X and satisfying the following condition:

(PS) Every sequence $\{x_n\} \subset X$ such that $\{f(x_n)\}$ is bounded and $\|f'(x_n)\| \rightarrow 0$ in X^* has a strongly convergent subsequence.

If there is an open neighborhood Ω of x_0 and a point $x_1 \notin \bar{\Omega}$ such that

$$f(x_0), f(x_1) < c_0 \leq \inf_{\partial\Omega} f,$$

let

$$\Gamma = \{g \in C([0, 1]; X) | g(0) = x_0, g(1) = x_1\}.$$

Then

$$c = \inf_{g \in \Gamma} \max_{t \in [0, 1]} f(g(t)) \geq c_0,$$

is a critical value of f : there is $\bar{x} \in X$ such that $f(\bar{x}) = c$ and $f'(\bar{x}) = 0$, where $f'(\bar{x})$ denotes the Frechet derivative of f at \bar{x} .

The generalization of the Mountain Pass Lemma of Ghoussoub-Preiss [7] involves the modification of the classical Palais-Smale condition:

Definition 1.2. Let X be a Banach space, and F is a closed subset of the Banach space X and ϕ a Gâteaux-differentiable functional on X . The $(PS)_{F,c}$ condition is the following: if $\{x_n\} \subset X$ is a sequence satisfying

(i).

$$d(x_n, F) \rightarrow 0.$$

where and in the following $d(x, F) = \inf_{y \in F} \|x - y\|$ denotes the distance between the point x and the set F .

(ii). $\phi(x_n) \rightarrow c$.

(iii). $\phi'(x_n) \rightarrow 0$.

Then $\{x_n\}$ has a strongly convergent subsequence.

Let $C^{1-0}(X; R)$ be the space of locally Lipschitz mappings from X to R , $\Phi \in C^{1-0}(X; R)$, we set (Clarke[4]):

$$\partial\Phi(x) = \{x^* \in X^* | \langle x^*, v \rangle \leq \Phi^0(x, v), \forall v \in X\}, \quad (1.1)$$

where and in the following,

$$\Phi^0(x, v) = \lim_{w \rightarrow x} \sup_{t \rightarrow 0^+} \frac{\Phi(w + tv) - \Phi(w)}{t}$$

denotes the generalized directional derivative of Φ at the point x along the direction v .

Definition 1.3. Let X be a Banach space, and $F \subset X$ a closed subset. We say $\Phi \in C^{1-0}(X; R)$ meets the $(CPS)_{F,c}$ condition if for all $\{x_n\} \subset X$ satisfying

- (1) $d(x_n, F) \rightarrow 0$,
- (2) $\Phi(x_n) \rightarrow c$,
- (3) $(1 + \|x_n\|) \cdot \min_{y^* \in \partial\Phi(x_n)} \|y^*\| \rightarrow 0$,

then $\{x_n\}$ has a convergent subsequence.

We can define δ distance ([6]):

$$\delta(x_1, x_2) = \inf\{l(c) \mid c \in C^1([0, 1], X), c(0) = x_1, c(1) = x_2\}, \quad (1.2)$$

here

$$l(c) = \int_0^1 \frac{\|\dot{c}(t)\|}{1 + \|c(t)\|} dt. \quad (1.3)$$

Then $\text{dist}_\delta(x, F) = \inf\{\delta(x, y) \mid y \in F\}$.

Definition 1.4. Let X be a Banach space, and $F \subset X$ a closed subset. We say $\Phi \in C^{1-0}(X; R)$ meets the $(CPS)_{F,c;\delta}$ condition if for all $\{x_n\} \subset X$ satisfying

- (1) $\text{dist}_\delta(x_n, F) \rightarrow 0$,
- (2) $\Phi(x_n) \rightarrow c$,
- (3) $(1 + \|x_n\|) \cdot \min_{y^* \in \partial\Phi(x_n)} \|y^*\| \rightarrow 0$,

then $\{x_n\}$ has a convergent subsequence.

We recall the Mountain-Pass Theorem generalized by Ghoussoub and Preiss[7] for a continuous and Gâteaux-differentiable functional with the $(PS)_{F,c}$ condition:

Theorem 1.5. ([7]) *Let $\varphi : X \rightarrow R$ be a continuous and Gâteaux-differentiable functional on a Banach space X such that $\varphi' : X \rightarrow X^*$ is continuous from the norm topology of X to the w^* -topology of X^* . Take $u, v \in X$, and let*

$$c = \inf_{g \in \Gamma} \max_{t \in [0,1]} \varphi(g(t))$$

where $\Gamma = \Gamma_u^v$ is the set of all continuous paths joining u and v . Suppose F is a closed subset of X such that $F \cap \{x \in X | \varphi(x) \geq c\}$ separates u and v , and φ satisfies $(PS)_{F,c}$ condition, then there exists a critical point $\bar{x} \in F$ for φ on F with critical value c :

$$\varphi(\bar{x}) = c, \varphi'(\bar{x}) = 0$$

A key ingredient in the proof of this theorem is provided by the following fundamental theorem in non-convex and nonlinear functional analysis established in the 1974 paper of Ivar Ekeland [5].

Theorem 1.6. ([5],) *Let (X, d) be a complete metric space with metric d , $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous functional not identically $+\infty$, and bounded from below. Let $\varepsilon > 0$ and $u \in X$ such that*

$$f(u) \leq \inf_{x \in X} f(x) + \varepsilon,$$

then for any given $\lambda > 0$, there exists $v_\lambda \in X$ such that $f(v_\lambda) \leq f(u)$, $d(u, v_\lambda) \leq \lambda$, and

$$f(w) > f(v_\lambda) - \frac{\varepsilon}{\lambda} d(v_\lambda, w), \quad \forall w \neq v_\lambda.$$

Ekeland's variational principle has found numerous applications; in particular, prior to Ghoussoub-Preiss[7], it was used by Shi [14] to prove a Mountain Pass Lemma and general min-max theorems for locally Lipschitz functionals(K.C.Chang[3]). In this paper, we will use Ekeland's variational principle to generalize the Ghoussoub-Preiss Theorem to the case of locally Lipschitz functional class C^{1-0} with $(CPS)_{F,c;\delta}$ or $(CPS)_{F,c}$.

Theorem 1.7. *Let X be a Banach space with the norm $||\cdot||$, $C^0([0, 1]; X)$ be the space of continuous mappings from $[0, 1]$ to X , and $\Phi : X \rightarrow \mathbb{R}$ be a locally Lipschitz functional. For $z_0, z_1 \in X$, define*

$$\begin{aligned} \Gamma &= \{c \in C^0([0, 1]; X) | c(0) = z_0, c(1) = z_1\}, \\ \gamma &= \inf_{c \in \Gamma} \max_{0 \leq t \leq 1} \Phi(c(t)), \end{aligned}$$

and set

$$\Phi_\gamma = \{x \in X | \Phi(x) \geq \gamma\}.$$

Suppose $F \subset X$ is a closed subset such that $F \cap \Phi_\gamma$ separates z_0 and z_1 , then there exists a sequence $\{x_n\} \subset X$ such that

$$\text{dist}_\delta(x_n, F) \rightarrow 0, \quad \Phi(x_n) \rightarrow \gamma, \quad \text{and} \quad (1 + \|x_n\|) \min_{y^* \in \partial\Phi(x_n)} \|y^*\| \rightarrow 0. \quad (1.4)$$

Theorem 1.8. *In the above Theorem 1.7, if we add the condition that the set F is norm-bounded in the Banach space X , then we have a sequence $\{x_n\} \subset X$ such that*

$$d(x_n, F) \rightarrow 0, \quad \Phi(x_n) \rightarrow \gamma, \quad \text{and} \quad (1 + \|x_n\|) \min_{y^* \in \partial\Phi(x_n)} \|y^*\| \rightarrow 0. \quad (1.5)$$

Theorem 1.9. *Under the assumptions of Theorem 1.7, if Φ satisfies $(CPS)_{F, \gamma; \delta}$ condition, then γ is a critical value for Φ :*

$$\Phi(\bar{x}) = \gamma, 0 \in \Phi'(\bar{x})$$

Theorem 1.10. *In the above Theorem 1.7, if we add the condition that the set F is bounded in the norm in the Banach space X , then we can change the $(CPS)_{F, \gamma; \delta}$ condition as $(CPS)_{F, \gamma}$ condition, and we have that there exists a critical point $\bar{x} \in F$ for Φ on F with critical value γ :*

$$\Phi(\bar{x}) = \gamma, 0 \in \Phi'(\bar{x})$$

2 The Proofs of Theorems 1.7-1.10

Proof. Since the main ingredient is still Ekeland's variational principle, we utilize some notations and ideas from [7] and [8], but we must deviate in a few key steps.

Since the closed set $F_\gamma = \Phi_\gamma \cap F$ separates z_0 and z_1 , we can write

$$X \setminus F_\gamma = \Omega_0 \bigcup \Omega_1$$

where $z_0 \in \Omega_0$, $z_1 \in \Omega_1$, for open sets Ω_0 and Ω_1 , Ω_0 and Ω_1 are disjoint.

Choose ε which satisfies

$$0 < \varepsilon < \frac{1}{2} \min\{1, \text{dist}_\delta(z_0, F_\gamma), \text{dist}_\delta(z_1, F_\gamma)\}. \quad (2.1)$$

By the definition of Γ , we can find $c \in \Gamma$ such that

$$\max_{0 \leq t \leq 1} \Phi(c(t)) < \gamma + \frac{\varepsilon^2}{4}. \quad (2.2)$$

If we define t_0 and t_1 by

$$\begin{aligned} t_0 &= \sup\{t \in [0, 1] | c(t) \in \Omega_0, \text{dist}_\delta(c(t), F_\gamma) \geq \varepsilon\}, \\ t_1 &= \inf\{t \in [t_0, 1] | c(t) \in \Omega_1, \text{dist}_\delta(c(t), F_\gamma) \geq \varepsilon\}, \end{aligned}$$

then since $c(0) = z_0$, hence by (2.1) and continuity of c , we have $t_0 > 0$. Moreover, by $c(t_0) \in \bar{\Omega}_0$ and $\text{dist}_\delta(c(t_0), F_\gamma) \geq \varepsilon$, we have $c(t_0) \in \Omega_0$. Then $\Omega_0 \cap \Omega_1 = \emptyset$ implies $t_1 > t_0$. By (2.1) and continuity of c , we have $t_1 < 1$. So we have $0 < t_0 < t_1 < 1$.

Let

$$\Gamma(t_0, t_1) = \{f \in C^0([t_0, t_1], X) | f(t_0) = c(t_0), f(t_1) = c(t_1)\}, \quad (2.3)$$

and consider the following distance in $\Gamma(t_0, t_1)$:

$$\rho(f_1, f_2) = \max_{t_0 \leq t \leq t_1} \delta(f_1(t), f_2(t)), \quad (2.4)$$

where

$$\delta(x_1, x_2) = \inf\{l(c) | c \in C^1([0, 1], X), c(0) = x_1, c(1) = x_2\}, \quad (2.5)$$

here

$$l(c) = \int_0^1 \frac{\|\dot{c}(t)\|}{1 + \|c(t)\|} dt. \quad (2.6)$$

For $x \in X$, we define function:

$$\Psi(x) = \max\{0, \varepsilon^2 - \varepsilon \text{dist}_\delta(x, F_\gamma)\}. \quad (2.7)$$

A map $\varphi : \Gamma(t_0, t_1) \rightarrow R$ is defined by

$$\varphi(f) = \max_{t_0 \leq t \leq t_1} \{\Phi(f(t)) + \Psi(f(t))\}. \quad (2.8)$$

Since

$$f(t_0) = c(t_0) \in \Omega_0, f(t_1) = c(t_1) \in \Omega_1, \quad (2.9)$$

there exists $t_f \in (t_0, t_1)$ satisfying $f(t_f) \in \partial\Omega_0 \subset F_\gamma$; therefore,

$$\text{dist}_\delta(f(t_f), F_\gamma) = 0, \quad (2.10)$$

and for $\forall f \in \Gamma(t_0, t_1)$, we have

$$\varphi(f) \geq \Phi(f(t_f)) + \Psi(f(t_f)) \geq \gamma + \varepsilon^2. \quad (2.11)$$

On the other hand, if we denote $\hat{c} = c|_{[t_0, t_1]}$, then

$$\varphi(\hat{c}) \leq \max_{0 \leq t \leq 1} \{\Phi(c(t)) + \Psi(c(t))\} \leq \gamma + \frac{5}{4}\varepsilon^2. \quad (2.12)$$

Notice that $\Gamma(t_0, t_1)$ is a complete metric space [5,6]; since Φ and Ψ are lower semi-continuous, so φ is lower semi-continuous; and (2.11) implies ϕ has lower bound; by (2.11) and (2.12), we have

$$\varphi(\hat{c}) \leq \inf \varphi + \frac{\varepsilon^2}{4}. \quad (2.13)$$

In Ekeland's variational principle, we use $\frac{\varepsilon^2}{4}$ in place of ϵ , and let $\lambda = \frac{\varepsilon}{2}$, then there exists $\hat{f} \in \Gamma(t_0, t_1)$, such that

$$\begin{aligned} \varphi(\hat{f}) &\leq \varphi(\hat{c}), \\ \rho(\hat{f}, \hat{c}) &\leq \frac{\varepsilon}{2}, \\ \varphi(f) &\geq \varphi(\hat{f}) - \frac{\varepsilon}{2}\rho(f, \hat{f}), \quad \forall f \in \Gamma(t_0, t_1). \end{aligned}$$

Let

$$M = \{t \in [t_0, t_1] | \Phi(\hat{f}(t)) + \Psi(\hat{f}(t)) = \varphi(\hat{f})\}. \quad (2.14)$$

The claim is that M is a non-empty compact set which avoids t_0 and t_1 .

By the definitions of t_0 and t_1 , we have

$$\text{dist}_\delta(c(t_i), F_\gamma) \geq \varepsilon, \quad i = 0, 1, \quad (2.15)$$

so $\Psi(\hat{c}(t_i)) = 0$ and by (2.2) and (2.11), we have

$$\Phi(\hat{f}(t_i)) + \Psi(\hat{f}(t_i)) \leq \Phi(\hat{c}(t_i)) + \Psi(\hat{c}(t_i)) \leq \gamma + \frac{\varepsilon^2}{4} < \varphi(\hat{f}), \quad i = 0, 1, \quad (2.16)$$

which implies $t_0, t_1 \notin M$.

We claim that there exists $t \in M$ such that

$$\min_{x^* \in \partial\Phi(\hat{f}(t))} \|x^*\|(1 + \|\hat{f}(t)\|) \leq \frac{3\varepsilon}{2}; \quad (2.17)$$

otherwise, for any $t \in M$,

$$\min_{x^* \in \partial\Phi(\hat{f}(t))} \|x^*\|(1 + \|\hat{f}(t)\|) > \frac{3\varepsilon}{2}. \quad (2.18)$$

It is well known that $\|x^*\| = \sup_{v \neq 0} \frac{\langle x^*, v \rangle}{\|v\|}$ where

$$x^* \in \partial\Phi(\hat{f}(t)) = \{x^* \in X^* | \langle x^*, v \rangle \leq \Phi^0(\hat{f}(t), v), \forall v \in X\}, \quad (2.19)$$

where and in the following,

$$\Phi^0(x, v) = \limsup_{w \rightarrow x} \sup_{t \rightarrow 0^+} \frac{\Phi(w + tv) - \Phi(w)}{t}$$

denotes the generalized directional derivative of Φ at the point x along the direction v . Notice that

$$\Phi^0(x, v) = \max\{\langle x^*, v \rangle \mid x^* \in \partial\Phi(x)\}. \quad (2.20)$$

Then for all $t \in M$, there exists $u(t) \in X$, s.t. $\|u(t)\| = (1 + \|\hat{f}(t)\|)$ and

$$\Phi^0(\hat{f}(t), u(t)) < -\frac{3\varepsilon}{2}. \quad (2.21)$$

Let $t \in M$ be such that (2.18) holds, then

$$\partial\Phi(\hat{f}(t)) \cap \frac{3\varepsilon}{2(1 + \|\hat{f}(t)\|)} B_{X^*} = \emptyset.$$

Notice that sets $\partial\Phi(\hat{f}(t))$ and $\frac{3\varepsilon}{2(1 + \|\hat{f}(t)\|)} B_{X^*}$ are convex and w^* compact, so by the separate Theorem, the two sets can be separated by an element of X , i.e., there is $v_0 \in X$ such that $\|v_0\| = 1$ and

$$\sup\{\langle x^*, v_0 \rangle \mid x^* \in \frac{3\varepsilon}{2(1 + \|\hat{f}(t)\|)} B_{X^*}\} < \inf\{\langle x^*, v_0 \rangle \mid x^* \in \partial\Phi(\hat{f}(t))\}$$

Notice that the left side of the above inequality is just $\frac{3\varepsilon}{2(1 + \|\hat{f}(t)\|)}$. Hence if we let $h = -(1 + \|\hat{f}(t)\|)v_0$, then we have

$$\sup\{\langle x^*, h \rangle \mid x^* \in \partial\Phi(\hat{f}(t))\} < -\frac{3\varepsilon}{2}.$$

Notice that the left side of the above inequality is equal to $\Phi^0(\hat{f}(t), h)$, we get (2.21) for $u(t) = h$. Let

$$N(t) = \{s \in M \mid \Phi^0(\hat{f}(s), u(t)) < -\frac{3\varepsilon}{2}\}$$

Since $x \mapsto \Phi^0(x, v)$ is upper semicontinuous for any given v , so $N(t)$ is an open subset of M , and M can be covered by the set of the open sets $N(t)$ when $t \in M$. Since M is a compact set, so we can pick the finite open sub-covers for M : $N(t_k)$, $0 \leq k \leq K$, then for the partition of unity associated with this cover on M , there are continuous functions $\xi_k(t) : 0 \leq \xi_k(t) \leq 1, 0 \leq k \leq K$, and $\sum_{k=1}^K \xi_k(t) = 1$.

Let $v(t) = \sum_{k=0}^K \xi_k(t)u(t_k)$, and observe the continuous map $v : M \rightarrow X$ satisfies

$$\Phi^0(\hat{f}(t), v(t)) < -\frac{3\varepsilon}{2}, \quad \|v(t)\| \leq \sum_{k=0}^K \xi_k(t)(1 + \|\hat{f}(t)\|) = (1 + \|\hat{f}(t)\|). \quad (2.22)$$

Since $M \subset [t_0, t_1]$, M is a nonempty compact set, and $t_0, t_1 \notin M$, so by Tietze extension Theorem, we can extend v as a continuous function defined on $[t_0, t_1]$ (which we still denote by v) which satisfies $v(t_0) = v(t_1) = 0$ and

$$\|v(t)\| \leq (1 + \|\hat{f}(t)\|), \forall t \in [t_0, t_1]. \quad (2.23)$$

Since $v(t_0) = v(t_1) = 0$, $\forall h > 0$, $\hat{f} + hv \in \Gamma$; hence,

$$\varphi(\hat{f} + hv) \geq \varphi(\hat{f}) - \frac{\varepsilon}{2} \rho(\hat{f} + hv, \hat{f}). \quad (2.24)$$

Choose $t_h \in [t_0, t_1]$ which satisfies:

$$\varphi(\hat{f} + hv) = (\Phi + \Psi)(\hat{f}(t_h) + hv(t_h)). \quad (2.25)$$

Notice that here t_h is defined for each $h > 0$. By the definition of φ , we know that for any $h > 0$, there holds

$$\varphi(\hat{f}) \geq (\Phi + \Psi)(\hat{f}(t_h)). \quad (2.26)$$

So $\forall h > 0$ we have

$$(\Phi + \Psi)(\hat{f}(t_h) + hv(t_h)) \geq (\Phi + \Psi)(\hat{f}(t_h)) - \frac{\varepsilon}{2} \rho(\hat{f} + hv, \hat{f}); \quad (2.27)$$

that is,

$$\Phi(\hat{f}(t_h) + hv(t_h)) - \Phi(\hat{f}(t_h)) \geq -\Psi(\hat{f}(t_h) + hv(t_h)) + \Psi(\hat{f}(t_h)) - \frac{\varepsilon}{2} \rho(\hat{f} + hv, \hat{f}). \quad (2.28)$$

If we recall the definition of Ψ , then Ψ is ε -Lipschitz, and so the above inequality implies

$$\Phi(\hat{f}(t_h) + hv(t_h)) - \Phi(\hat{f}(t_h)) \geq -\frac{3\varepsilon}{2} \rho(\hat{f} + hv, \hat{f}). \quad (2.29)$$

Notice that if $h_n \rightarrow 0^+$, we can pass to a sequence of $\{t_{h_n}\}$ such that $t_{h_n} \rightarrow \tau \in M$ since M is compact, we have

$$\limsup_{n \rightarrow +\infty} \frac{\Phi(\hat{f}(t_{h_n}) + h_n v(t_{h_n})) - \Phi(\hat{f}(t_{h_n}))}{h_n} \geq -\frac{3\varepsilon}{2} \liminf_{n \rightarrow +\infty} \frac{\rho(\hat{f} + h_n v, \hat{f})}{h_n}. \quad (2.30)$$

and further by $\Phi \in C^{1-0}$ and the definitions of Clark's generalized gradient and the metric ρ , we have

$$\Phi^0(\hat{f}(\tau), v(\tau)) \geq -\frac{3\varepsilon}{2} \max_{t_0 \leq t \leq t_1} \left(\frac{\|v(t)\|}{1 + \|\hat{f}(t)\|} \right) \geq -\frac{3\varepsilon}{2}, \quad (2.31)$$

In fact, by $\Phi \in C^{1-0}$ and the continuity for $v(t)$, we know that

$$\frac{\Phi(\hat{f}(t_{h_n}) + h_n v(t_{h_n})) - \Phi(\hat{f}(t_{h_n}) + h_n v(\tau))}{h_n} \leq L|v(t_{h_n}) - v(\tau)| \rightarrow 0,$$

hence

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} \frac{\Phi(\hat{f}(t_{h_n}) + h_n v(t_{h_n})) - \Phi(\hat{f}(t_{h_n}))}{h_n} \\ & \leq \limsup_{n \rightarrow +\infty} \frac{\Phi(\hat{f}(t_{h_n}) + h_n v(t_{h_n})) - \Phi(\hat{f}(t_{h_n}) + h_n v(\tau))}{h_n} \\ & \quad + \limsup_{n \rightarrow +\infty} \frac{\Phi(\hat{f}(t_{h_n}) + h_n v(\tau)) - \Phi(\hat{f}(t_{h_n}))}{h_n} \\ & = \Phi^0(\hat{f}(\tau), v(\tau)) \end{aligned}$$

We use the definition (2.4) of the metric ρ , we have

$$\rho(\hat{f} + h_n v, \hat{f}) = \max_{t_0 \leq t \leq t_1} \inf \left\{ \int_0^1 \frac{\|\dot{c}(s)\|}{1 + \|c(s)\|} ds, c(s) \in C^0([0, 1], X), c(0) = \hat{f}, c(1) = \hat{f} + h_n v \right\}.$$

Specially, we take the following loop connecting \hat{f} and $\hat{f} + h_n v$:

$$c(s) = (1 - s)\hat{f}(t) + s(\hat{f}(t) + h_n v(t)),$$

then we have

$$\dot{c}(s) = h_n v(t),$$

and

$$c(s) = (1 - s)\hat{f}(t) + s(\hat{f}(t) + h_n v(t)) \rightarrow \hat{f}(t), n \rightarrow +\infty.$$

So we have that

$$\liminf_{n \rightarrow +\infty} \frac{\rho(\hat{f} + h_n v, \hat{f})}{h_n} \leq \max_{t_0 \leq t \leq t_1} \left(\frac{\|v(t)\|}{1 + \|\hat{f}(t)\|} \right),$$

and (2.31) is proved, which violates (2.22) and shows that we cannot have the inequality (2.18); therefore, there is $\bar{t} \in M$ such that

$$\min_{x^* \in \partial \Phi(\hat{f}(\bar{t}))} \|x^*\| (1 + \|\hat{f}(\bar{t})\|) \leq \frac{3\varepsilon}{2}. \quad (2.32)$$

By the definitions of t_0 and t_1 , we have that $d(\hat{c}(t), F_\gamma) \leq \epsilon$ for $t_0 < t < t_1$, furthermore, by the continuous properties of $\hat{c}(t)$ and $d(x, F_\gamma)$ on x , we have that

$$d(\hat{c}(t), F_\gamma) \leq \epsilon, \forall t \in [t_0, t_1].$$

Notice that here $d(\hat{c}(t), F_\gamma)$ is the distance between $\hat{c}(t)$ and F_γ deduced by the norm in the Banach space X . We use the notation $dis_\delta(\hat{c}(t), F_\gamma)$ to denote the distance between $\hat{c}(t)$ and F_γ deduced by δ in (2.5). By the definitions of δ and the norm, we have that

$$\delta(x_1, x_2) \leq \|x_1 - x_2\|$$

So

$$dis_\delta(\hat{c}(t), F_\gamma) \leq d(\hat{c}(t), F_\gamma) \leq \epsilon, \forall t \in [t_0, t_1].$$

We notice that ρ is the distance deduced by δ in (2.5), since $\rho(\hat{f}, \hat{c}) \leq \frac{\epsilon}{2}$, the triangle inequality implies that for all $t \in [t_0, t_1]$, we have

$$dis_\delta(\hat{f}(t), F_\gamma) \leq \frac{\epsilon}{2} + dis_\delta(\hat{c}(t), F_\gamma) \leq \frac{\epsilon}{2} + \epsilon = \frac{3\epsilon}{2}. \quad (2.33)$$

Set $x = \hat{f}(\bar{t})$, we get

$$dis_\delta(x, F_\gamma) \leq \frac{3\epsilon}{2};$$

If F is bounded and closed subset of X , then by the definition of δ , we know that ([6]) δ distance is equivalent to the norm distance, so there is $c > 0$ such that

$$dis_\delta(x, F_\gamma) \geq cd(x, F_\gamma)$$

Then $\varphi(\hat{f}) \leq \varphi(\hat{c})$ yields

$$\gamma + \epsilon^2 \leq \Phi(\hat{f}(\bar{t})) + \Psi(\hat{f}(\bar{t})) \leq \gamma + \frac{5\epsilon^2}{4}. \quad (2.34)$$

Then we get

$$\min_{x^* \in \partial\Phi(x)} \|x^*\| (1 + \|x\|) \leq \frac{3\epsilon}{2},$$

$$d(x, F_\gamma) \leq \frac{1}{c} \frac{3\epsilon}{2},$$

$$\gamma \leq \Phi(x) \leq \gamma + \frac{5\epsilon^2}{4}.$$

If we let $\epsilon = \frac{1}{n} \rightarrow 0$, then we arrive at a sequence $\{x_n\}$ which satisfies the requirements of Theorems 1.7 and 1.8, then Theorems 1.9 and 1.10 follow from Theorems 1.7 and 1.8. \square

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